CHAPTER 5

Discrete Random Variables

In some books there is a feeling of transition between the chapter discussing probability and the chapter discussing random variables. This should not be the case. The subject of random variables is nothing more than an extension of the basic probability we studied in the previous chapter, although there are a few new notions. We will also study three probability situations that are so common as to merit the presentation of standard formulas to handle them.

5.1 What Is a Random Variable?

Let’s return to a simple example from the previous section. Recall that the sample space for the tossing of a single coin is $S = \{H, T\}$. The study of probability is, of course, a branch of mathematics. Mathematical activities are somewhat curtailed in this example because we are speaking of events (“heads” and “tails”) that have nothing to do with numbers. It is the role of the random variable to remedy this situation.
Returning to the coin toss example, one might assign the number 0 to a head and the number 1 to a tail. These assignments are arbitrary and other assignments might be made.

**Example 5.1:** Two fair dice are tossed. The random variable $X$ is defined to be the sum of the dots facing up on the two dice. This is a common way to assign numbers to all the outcomes possible when two dice are tossed. This example can be extended by introducing the notion of the probability function.

A probability function is a rule that assigns probabilities to each of the possible outcomes of an experiment.

Both the random variable and its associated probability function can be conveniently represented in tabular form:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1/36</td>
</tr>
<tr>
<td>3</td>
<td>2/36</td>
</tr>
<tr>
<td>4</td>
<td>3/36</td>
</tr>
<tr>
<td>5</td>
<td>4/36</td>
</tr>
<tr>
<td>6</td>
<td>5/36</td>
</tr>
<tr>
<td>7</td>
<td>6/36</td>
</tr>
<tr>
<td>8</td>
<td>5/36</td>
</tr>
<tr>
<td>9</td>
<td>4/36</td>
</tr>
<tr>
<td>10</td>
<td>3/36</td>
</tr>
<tr>
<td>11</td>
<td>2/36</td>
</tr>
<tr>
<td>12</td>
<td>1/36</td>
</tr>
</tbody>
</table>

$\sum P(X) = 1$
The following schematic illustrates what is going on with the random variable and probability rule. The experiment takes place in the “real world.” The random variable replaces these actual physical outcomes with numbers, while the probability rule then gives the appropriate probabilities for each value of the random variable.

**Figure 5.1**
Schematic of the Function of a Random Variable

The random variable $X$ represents the number of heads obtained. List the random variable in tabular form along with its associated probability function.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/8</td>
</tr>
<tr>
<td>1</td>
<td>3/8</td>
</tr>
<tr>
<td>2</td>
<td>3/8</td>
</tr>
<tr>
<td>3</td>
<td>1/8</td>
</tr>
</tbody>
</table>

**Example 5.2:** Three coins are tossed. The random variable $X$ represents the number of heads obtained. List the random variable in tabular form along with its associated probability function.

A Look Back: Remember the Fundamental Principle of Counting from chapter four.

If you are not sure where the probabilities came from, recall that $n(S) = 2^3 = 8$. Then, count the number of ways to obtain 0, 1, 2, and 3 heads.

**Example 5.3:** Joe Cool lives in a city that has 8 traffic lights on Main Street. Let $X$ be the random variable that represents the number of lights that Joe gets through on a single pass through town without having to stop. (Assume that Joe is a good driver and does not “run” lights.) What are the possible values of the random variable $X$? Is it possible to give the probability function with the information given?
$X$ can take on the values 0, 1, 2, …, 8. It is not possible to specify the probability function without additional information. In fact, quite a bit of additional information is necessary to answer this question. In some cities the lights are synchronized so that if you make one you will make the rest if you drive the speed limit. In some cities the lights seem to be synchronized so that you cannot make more than one or two in a row. In some cities the lights seem to operate at random. To arrive at probabilities one would have to know about synchronization as well as the amount of time that each light is red, yellow and green. We won’t pursue this question any further, but it is a good example of how difficult a seemingly simple probability question can be.

Example 5.4: A logger is cutting down a tree. Let $X$ represent the length of the log that is obtained when the tree is felled. What possible values can $X$ have? What are the probabilities associated with the various lengths?

Perhaps this sounds like some kind of trick question to you. If so, this is a good sign. This example is fundamentally different from the three that have preceded it. The difference is that length is not a variable that can be counted. It can only be measured. The measured length can take on any value between 0 and whatever the length of the tallest possible tree can be. (We might say 1,000 ft just to be on the safe side.) As for the probabilities, this is a very difficult issue. Since there are an uncountably infinite number of possible tree lengths we will need a very special way to describe the probabilities associated with the lengths. That will be the subject of chapter six.

The difference between quantities that can be measured and those that can be counted is so fundamental that it merits a formal definition.
A quantity is said to be \textit{discrete} if it can be or conceivably could be counted. A quantity that can only be measured and takes on too many values to be counted is said to be \textit{continuous}. Such quantities can take on all values in a range of numbers.

Some people think that countable means finite. The set of natural numbers \{1, 2, 3, \ldots\} is infinite but is countable, and hence discrete. This is the reason for the phrase “or conceivably could be counted” in the definition. If one could count forever then the natural numbers could be counted. Another way to understand discrete quantities is to realize that they are distinct and separate. Each natural number is distinct and separate from the others. In the case of continuous quantities such as the length of the log, however, the possible lengths are so close together as to be indistinguishable from one another. This is why we can’t count the number of possible log lengths.

There is one more thing that needs to be said. All human endeavor is really discrete. While we recognize that in theory the log could be any one of an innumerable number of lengths, in practice we are going to measure it to the nearest foot or the nearest inch, or to the nearest of whatever unit we choose. As soon as we measure to the nearest anything we have made our measurement discrete, for you can count inches, pounds, centimeters or whatever. It is only when we speak theoretically that we can talk about things being continuous. In future chapters we will use continuous models to describe our work, but such models are only approximations of our discrete measurements. However, they will prove to be very useful.

\textbf{Example 5.5}: Determine whether each of the following variables is discrete or continuous.

a. The number of sales made by a regional salesman.

b. The dollar value of the sales made by the regional salesman.

c. The weight of gold nuggets found at a gold mine.

d. The weight of the cereal in a randomly selected box of corn flakes.
e. The number of successful oil wells that will be drilled by an oil exploration company next year.

Both parts a. and e. are clearly discrete (anything that starts with “The number” is discrete). Parts c. and d. are quite clearly continuous. Part b. is a bit tough to call. Technically, anything having to do with money is discrete because you can certainly count every possible monetary amount. Practically speaking, however, this situation would probably be modeled with continuous models. This is because the dollar amounts, while discrete, are so numerous as to be satisfactorily approximated by continuous models.

Finally, it should be noted that any discrete random variable and its accompanying probability function must satisfy all the rules for probabilities established in the last chapter. The following box stresses this fact.

**Example 5.6:** Consider each of the following random variables and their accompanying probability functions. Decide whether or not the probability functions are legitimate.

a.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1/5$</td>
</tr>
<tr>
<td>1</td>
<td>$1/10$</td>
</tr>
<tr>
<td>2</td>
<td>$3/10$</td>
</tr>
<tr>
<td>3</td>
<td>$3/10$</td>
</tr>
</tbody>
</table>
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b.  

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4/8</td>
</tr>
<tr>
<td>1</td>
<td>3/8</td>
</tr>
<tr>
<td>2</td>
<td>4/8</td>
</tr>
<tr>
<td>3</td>
<td>-1/8</td>
</tr>
</tbody>
</table>

c.  $P(X) = \frac{X}{10}, X = 1, 2, 3, 4$.

The random variable and corresponding probability function defined in part a. are not legitimate since the probabilities add up to 9/10 rather than 1.

The random variable and corresponding probability function defined in part b are not legitimate either since $P(X)$ for $X = 3$ is listed as a negative number, which of course is nonsense.

The random variable and associated probability function defined in part c is a legitimate random variable. This can be seen by filling out a random variable table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/10</td>
</tr>
<tr>
<td>2</td>
<td>2/10</td>
</tr>
<tr>
<td>3</td>
<td>3/10</td>
</tr>
<tr>
<td>4</td>
<td>4/10</td>
</tr>
</tbody>
</table>

The probabilities are all non-negative and sum to one as they must.

Exercises

Basic Practice

1. A single fair die is tossed. Let the random variable $X$ represent the number of spots showing on the side that lands face up. List the possible values of the random variable and also list the values of its probability function.
2. A couple plans to have four children. Let $X$ represent the number of girls. List the possible values of $X$ and also list the values of its probability function. Assume that male and female children are equally likely.

3. Is the function $P(X) = \frac{6 - |X - 7|}{36}$, $X = 2, 3, \ldots, 12$ a legitimate probability function? What does it remind you of?

4. Does the following table describe a legitimate random variable/probability function combination? Why, or why not?

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/10</td>
</tr>
<tr>
<td>2</td>
<td>2/10</td>
</tr>
<tr>
<td>3</td>
<td>3/10</td>
</tr>
<tr>
<td>4</td>
<td>3/10</td>
</tr>
</tbody>
</table>

5. Does the following table describe a legitimate random variable/probability function combination? Why, or why not?

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/10</td>
</tr>
<tr>
<td>2</td>
<td>3/10</td>
</tr>
<tr>
<td>3</td>
<td>-3/10</td>
</tr>
<tr>
<td>4</td>
<td>9/10</td>
</tr>
</tbody>
</table>

**Concepts**

6. How might an investor go about defining a random variable that would describe the performance of a $10,000 investment? What sources of information would he need to investigate in order to assign meaningful probabilities to the various possible outcomes? Where would he go to find out what the various possible outcomes might be? Do you think he will be able to give exact answers to these questions?

7. Classify each of the following random variables as discrete or continuous.
   a. The number of sales made by a used car salesman in a week.
   b. The weight of a randomly chosen person.
c. The diameter of a randomly chosen ball bearing from an assembly line.
d. The number of complete lines typed by a typist in a randomly chosen minute.

8. Classify each of the following random variables as discrete or continuous.
   a. The weight of a randomly chosen calf in a cattle weight gain study.
   b. The number of votes senator Jones will get in the next election.
   c. The time it will take for a randomly chosen rat to solve a maze.
   d. The temperature in Fahrenheit at 12:30 PM tomorrow.

5.2 The Mean and Variance of Discrete Random Variables

To introduce this section let us consider two discrete random variables of a type that can arise in business. Perhaps what is presented here is an oversimplification of the real world, but nonetheless it should suffice to convince you that what you are learning has practical value.

**Example 5.7:** An investor has $10,000 to invest. Two investment opportunities arise. The investor has done considerable research on the two investments and has arrived at the following probability assessments of the two investments. She believes that there is a 0.2 chance that investment $A$ will earn no money at all. She believes that there is a 0.5 chance that investment $A$ will earn $400 and a 0.3 chance that the investment will earn $1,000. Let’s call the amount of money earned by investment $A$ in one year $X$. Then, investment $A$ can be summarized as follows:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>0.2</td>
</tr>
<tr>
<td>$400$</td>
<td>0.5</td>
</tr>
<tr>
<td>$1,000$</td>
<td>0.3</td>
</tr>
</tbody>
</table>
The Mean and Variance of Discrete Random Variables

The probabilities of the yield for investment $B$ (denoted by $Y$) may be summarized as follows:

<table>
<thead>
<tr>
<th>$Y$</th>
<th>$P(Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1,000$</td>
<td>0.2</td>
</tr>
<tr>
<td>$400$</td>
<td>0.5</td>
</tr>
<tr>
<td>$2,000$</td>
<td>0.3</td>
</tr>
</tbody>
</table>

This means that $1,000 will be lost.

The natural question arises, “which investment is preferable?” To a large degree the answer to this question depends on the temperament of the person making the decision (as well as on how much money the person has to lose). However, before we can make any kind of a rational decision we need to analyze things carefully.

One way to do a careful analysis is to think in terms of what would happen “in the long run” if we made “many” such investments. To illustrate this type of thinking let’s suppose that we made investment $A$ ten times. What could we expect to happen? If everything went exactly according to the probabilities, we would expect to earn $0$ twice, $400$ five times and $1,000$ three times. If we were to treat these ten expected earnings as a population we would have:

$0, 0, 400, 400, 400, 400, 400, 1000, 1000, 1000$

Note: If you’re working through the example as you read, remember to use the calculator key for population standard deviation.

Some calculations (best done with a calculator) would reveal:

\[ \mu = 500 \text{ and } \sigma = 360.56. \]

What do these two numbers mean? The mean $\mu$ of 500 simply means that in the long run we would expect to average $500 earnings on our investment of $10,000. This is a 5% return on our investment. The standard deviation $\sigma$ is a measure of how much variation from the expected earnings of $500 is typical.

If we also made investment $B$ ten times, what could we expect to happen? If everything went exactly according to the probabilities we could expect to lose $1,000 twice, earn $400 five times, and earn $2,000 three times. Treating these ten expected earnings as a population we would have:
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-1000, -1000, 400, 400, 400, 400, 400, 2000, 2000, 2000

Some calculations (best done with a calculator) would reveal:

\[ \mu = 600 \text{ and } \sigma = 1058.30. \]

Now, with this added information, what would you do? In the long run investment B will yield higher amounts (an average of $600 per investment compared to an average of $500 per investment for investment A). However, investment B is also less reliable than investment A with a much higher standard deviation. Which investment is best? The answer depends on how averse you are to risk. Investment B has the highest expected yield, but it also has the greatest risk. The choice of investment depends on the temperament of the person making the investment.

The preceding example does serve to show how much value looking at long run expected values can have. Knowing the expected value (mean) and standard deviation of the returns on the investments gave us some good information that was not apparent just by looking at the possibilities. The importance of the mean and standard deviation of a random variable will grow as the complexity of the situation grows. With investments having hundreds of possible outcomes it is not at all possible to even talk about which one is best just by trying to wrap your mind around all the possibilities. Once again, the mean and standard deviation allow us to condense a lot of information into just two highly informative numbers.

The mean and standard deviation of a random variable are defined by:

\[ \mu = \sum X P(X) \quad (5.1) \]

The mean \( \mu \) of a random variable is also sometimes called the expected value, and denoted by \( E(X) \).

\[ \sigma = \sqrt{\sum (X - \mu)^2 P(X)} \quad (5.2) \]
You may be wondering why the equations don’t look exactly like the equations for mean and standard deviation in chapter two. These formulas really are the same as the ones in chapter two. They look different because of the context. The following example will help demonstrate the equivalence of these formulas with the ones for population mean and variance in chapter two.

**Example 5.8:** Consider the random variable $X$ defined by the following probability table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Thinking “in the long run” let’s consider again what we would expect to happen if we repeated the experiment defined by the random variable $X$ ten times. We would expect (if the outcomes followed the probabilities exactly) to get the following results (not necessarily in this order):

1, 1, 1, 2, 3, 3, 3, 3, 3, 3

Just treating these as a ten number population we would find:

\[
\mu = \frac{\sum X}{N} = \frac{1 + 1 + 1 + 2 + 3 + 3 + 3 + 3 + 3 + 3}{10}
\]

\[
= \frac{1(3) + 2(1) + 3(6)}{10}
\]

\[
= 1\left(\frac{3}{10}\right) + 2\left(\frac{1}{10}\right) + 3\left(\frac{6}{10}\right)
\]

\[
= 1 \times P(1) + 2 \times P(2) + 3 \times P(3)
\]

\[
= \sum XP(X).
\]

The same equivalence exists between the formulas

\[
\sigma^2 = \sum (X - \mu)^2 P(X) \quad \text{and} \quad \sigma^2 = \frac{\sum (X - \mu)^2}{N}
\]
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$P(X)$ serves the same role as dividing by $N$.

**Note:** All this talk of formulas has been presented for two reasons. First, to try to help you have a concept of why these formulas measure what they are supposed to, and second, to show you that they really are equivalent to the formulas in chapter two.

**A Look Back:** Remember that $\mu$ and $\sigma$ have represented the mean and standard deviation of a population. The symbols now do double duty to represent the mean and standard deviation of a random variable. However, example 5.8 should convince you that the two concepts are really one and the same.

Your calculator can once again make your life simple. Modern calculators are capable of entering the data pertinent to a random variable and automatically calculating the mean and standard deviation or variance for you. As usual, it is more important for you to know what $\mu$ and $\sigma$ measure and how to interpret them than it is to know how to calculate them. However, with calculators their calculation is so easy that there is no reason you shouldn’t know how to find them as well.

Note that the mean and variance of a random variable are always represented by the Greek letters $\mu$ and $\sigma^2$ because a random variable is considered to be a theoretical thing (as is a population) and is in no way derived from a sample.

**Example 5.9:** This example is provided entirely for those who do not have a “smart” calculator available. What you really ought to do if at all possible is buy a new calculator if yours won’t do this, but if you can’t, this example is provided to show you how to do this type of exercise “by hand.”

Find the mean and variance of the following random variable:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
<th>$XP(X)$</th>
<th>$(X - \mu)^2$</th>
<th>$(X - \mu)^2 P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.2</td>
<td>$(1 - 2.3)^2 = 1.69$</td>
<td>$1.69(0.2) = 0.338$</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.6</td>
<td>$(2 - 2.3)^2 = 0.09$</td>
<td>$0.09(0.3) = 0.027$</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>1.5</td>
<td>$(3 - 2.3)^2 = 0.49$</td>
<td>$0.49(0.5) = 0.245$</td>
</tr>
</tbody>
</table>

---

$\mu = 2.3$  $\sigma^2 = 0.61$

The heavily outlined box defines the random variable. The rest of the table is for the calculations necessary to solve the problem. The first thing you must do is add a column to the table containing $XP(X)$. The sum of this column is $\mu = 2.3$. Next, another column for $(X - \mu)^2$ is added. Finally, a column for $(X - \mu)^2 P(X)$ is added. The total of this column is $\sigma^2 = 0.61$. The standard deviation is $\sigma = \sqrt{0.61} = 0.78102497$. 

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Calculations Involving Grouped Data

A subject highly related to that of calculating means and standard deviations for random variables is the subject of calculating means and standard deviations for grouped data.

**Example 5.10:** The following table gives a frequency distribution for the results of a sample of American families. \( X \) represents the number of children in the family. The frequencies tell how many of the sampled families had that many children.

<table>
<thead>
<tr>
<th>( X )</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>1</td>
<td>32</td>
</tr>
<tr>
<td>2</td>
<td>58</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

Your calculator can be used to work this exercise. Note that your calculator provides a way for you to treat this as either a sample or a population. The mean and variance for this example are:

\[
\bar{X} = 2.11, \quad s^2 = 1.87
\]

After spending a few moments with your calculator manual you should return here and see if you can duplicate these results.

**Example 5.11:** Grouped data often arise from histograms and/or their accompanying frequency distributions. Sometimes you may find that a frequency distribution is available to you, but the actual data are not. In this case you can still approximate the mean and the standard deviation.
To deal with this situation we must realize that while the classes do not give us the actual data we should be able to get a close approximation of the mean or standard deviation. We accomplish this by doing the common sense thing and treating all data in a class as though it fell in the middle of the class (called the class mark). In other words we treat the data as though it were the same as:

$$\bar{X} = \frac{\sum Xf}{\sum f}$$

We then arrive at $\bar{X} = 13.18$ and $s = 6.83$ (treating the data as a sample). If the data were a census we would obtain $\mu = 13.18$ and $\sigma = 6.79$.

QwikStat can also find the mean and variance of random variables or variables defined by frequency tables. To illustrate we will present the QwikStat solution to example 5.8. The datasheet will look like this:

<table>
<thead>
<tr>
<th>Class</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 5</td>
<td>10</td>
</tr>
<tr>
<td>5 to 10</td>
<td>18</td>
</tr>
<tr>
<td>10 to 15</td>
<td>32</td>
</tr>
<tr>
<td>15 to 20</td>
<td>12</td>
</tr>
<tr>
<td>20 to 25</td>
<td>10</td>
</tr>
<tr>
<td>25 to 30</td>
<td>6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>10</td>
</tr>
<tr>
<td>7.5</td>
<td>18</td>
</tr>
<tr>
<td>12.5</td>
<td>32</td>
</tr>
<tr>
<td>17.5</td>
<td>12</td>
</tr>
<tr>
<td>22.5</td>
<td>10</td>
</tr>
<tr>
<td>27.5</td>
<td>6</td>
</tr>
</tbody>
</table>
**Computer Example:**

Choosing Prob/Random Variables will get you the answers of $\mu = 2.3$ and $\sigma = 0.9$. Again, there is a shortcut. If you will highlight the X and P(X) columns before the menu choice, then Qwik-Stat will automatically know what to do. This shortcut only works, however, if you put the random variable first, followed by the probabilities.

The results are as follows:

<table>
<thead>
<tr>
<th>Var</th>
<th>X</th>
<th>P(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.5</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17:36:55</td>
</tr>
<tr>
<td>2</td>
<td>X weighted by P(X)</td>
</tr>
<tr>
<td>3</td>
<td>Mean</td>
</tr>
<tr>
<td>4</td>
<td>Std. Dev.</td>
</tr>
<tr>
<td>5</td>
<td>Variance</td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>
Exercises

Basic Practice

9. Find the mean, variance and standard deviation of the random variable defined by:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
</tr>
<tr>
<td>4</td>
<td>0.3</td>
</tr>
<tr>
<td>5</td>
<td>0.1</td>
</tr>
</tbody>
</table>

10. Find the mean, variance and standard deviation of the random variable defined by:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
</tr>
<tr>
<td>15</td>
<td>0.1</td>
</tr>
</tbody>
</table>

11. Find the mean, variance and standard deviation of the random variable defined by:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.2</td>
</tr>
<tr>
<td>-1</td>
<td>0.2</td>
</tr>
<tr>
<td>0</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
</tr>
</tbody>
</table>

12. Find the mean, variance and standard deviation of the random variable defined by:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>0.1</td>
</tr>
<tr>
<td>1100</td>
<td>0.2</td>
</tr>
</tbody>
</table>
13. In each of the following tables, either fill in the missing entry to give a legitimate probability distribution for the random variable or indicate why you can’t.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.6</td>
<td>5</td>
<td>0.3</td>
</tr>
<tr>
<td>5</td>
<td>0.8</td>
<td>8</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Also, find the mean $\mu$ and the standard deviation $\sigma$ for those random variables that are legitimate.

14. In each of the following tables, either fill in the missing entry to give a legitimate probability distribution for the random variable or indicate why you can’t.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>5</td>
<td>0.3</td>
</tr>
<tr>
<td>5</td>
<td>0.8</td>
<td>8</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Also, find the mean $\mu$ and the standard deviation $\sigma$ for those random variables that are legitimate.

15. In each of the following tables, either fill in the missing entry to give a legitimate probability distribution for the random variable or indicate why you can’t.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$P(X)$</th>
<th>$X$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>5</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0.8</td>
<td>10</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Also, find the mean $\mu$ and the standard deviation $\sigma$ for those random variables that are legitimate.
Concepts

16. How would you go about defining a random variable that describes your chances of coming to a randomly approached traffic signal on red, yellow, or green? What factors might influence the probabilities that you will come to the light during each of the three colors?

17. How would you go about defining a random variable that describes your chances of waking up before your alarm clock goes off, during the ringing, or after your alarm clock has given up on you?

Applied

18. A one year business investment of $1,000,000 is defined by the following random variable.

<table>
<thead>
<tr>
<th>Interest Earned</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0</td>
<td>0.2</td>
</tr>
<tr>
<td>$10</td>
<td>0.1</td>
</tr>
<tr>
<td>$100</td>
<td>0.2</td>
</tr>
<tr>
<td>$1,000</td>
<td>0.1</td>
</tr>
<tr>
<td>$10,000</td>
<td>0.2</td>
</tr>
<tr>
<td>$100,000</td>
<td>0.2</td>
</tr>
</tbody>
</table>

a. Find $\mu$ and $\sigma$.

b. Discuss the investment. If you were in a position to do so, would you choose to make this investment? Why, or why not?

19. Suppose a small clinic has recorded the number of patients per day that they have admitted to local hospitals for a random sample of days chosen from all business days of the last five years. Compute the mean and standard deviation for the number of patients per day who are admitted to the hospital given the data below.
In the above table the frequency represents the number of days in the sample that had the number of patients admitted indicated in the column to the left.

20. Suppose a student has applied to three different graduate schools for admission. The random variable $X$ represents the number of schools that accept her.

- What is the probability that no schools will accept the student?
- What is the probability that at least two schools will accept her?
- What is the expected number of schools that will accept her?
- What is the standard deviation of the number of schools that will accept her?

### Two Counting Formulas

The two formulas presented in this section may be viewed in two ways. One approach is to point out the formulas, what they’re good for, and how to get your calculator to do them and move on. The
other approach is to take a few minutes to really try to understand how the formulas work. Explanations follow that should suffice to not only drop the formulas into your lap, but to also explain where they come from and why they work.

Suppose you are playing a revised version of musical chairs with six people and four chairs. In this version of the game people go about their business at a party until some predetermined signal is given, at which time they all rush to sit down. There is no seating order specified. The question is, “in how many ways can four of the six people seat themselves in the four chairs?”

There are six ways to fill the first chair. After one person is seated there are five people left as possibilities to sit in the second chair, then four, then three to fill the last chair. According to the Fundamental Principle of Counting from the last chapter, to find the total number of ways that the four tasks of filling the four chairs can be done is found by multiplying the four numbers together. Thus, the number of ways the four chairs can be filled is $6 \cdot 5 \cdot 4 \cdot 3 = 360$.

There is a way to put this into a formula. To see how to do this we must first refresh your memory (for some of you this may be new) about factorials.
Returning to our example of the four chairs, notice that the number of ways four people can be chosen from six and seated in four chairs is:

\[ \frac{6!}{2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 1}{2 \cdot 1} = 6 \cdot 5 \cdot 3. \]

These observations lead us to a formula for the number of ways we can arrange \( r \) objects taken from \( n \) distinct objects. This type of number is referred to as a \textit{permutation}.

The number of \textit{permutations} of \( r \) objects chosen from \( n \) distinct objects is given by:

\[ P(n, r) = \frac{n!}{(n-r)!} \]

The symbols \( P(n, r) \) and \( n^r \) mean exactly the same thing. They both mean “the number of ways to arrange \( r \) objects taken from \( n \) distinct objects.”

While permutations are important in their own right, for our purposes permutations are a stepping stone to \textit{combinations}. The word permutation carries the connotation of order, while the word combination refers only to which items are selected. When counting permutations we considered not only \textit{who} was seated, but \textit{which chair} they were seated in as well. This gives rise to an important question. Once the four people who are to be seated have been chosen, how many different orders can they sit in? To see, let’s once again...
visualize our four chairs. The difference this time is that our four people have already been chosen. We just want to count the number of ways that we could arrange the four people in the four chairs.

There are four people available to sit in the first chair, three in the next, and so on. Thus, there are $4! = 4 \cdot 3 \cdot 2 \cdot 1$ ways to seat four people in four chairs. This answer is very important to us because it tells us precisely how many times each combination is counted as a permutation. So, we come to the final question, “how many ways can four people be chosen from six?” We know that there are $\binom{6}{4}$ ways to select and order four people chosen from 6. We also know that there are 4! ways to order four people. Thus, the permutation counts each group of four people 4! times (a combination should count it only once). If we divide the permutation by 4! we will divide out the multiple counts and have the combination. The number of ways four people can be chosen from six without regard to order is:

$$\frac{6!}{4!2!}$$
If we reason in general we arrive at the following formula for combinations:

\[ C(n, r) = \frac{n!}{r!(n-r)!} \]  

Before you panic and say “I didn’t understand any of that!” you should know that your calculator will once again come to the rescue. You will be a richer person if you take time to understand the explanation of combinations and permutations above. You will also have a better chance of knowing when to use these formulas if you have followed their explanation. However, if you learn to use your calculator well, and know when to use it, you can survive without understanding the above explanation.

It helps many students when trying to distinguish between permutations and combinations to remember that there are more permutations than combinations for any given \( n \) and \( r \). This is because permutations are concerned not only with which objects were chosen, but also with the various orderings of the objects.

**Example 5.12:** Calculate:

a. \( \binom{6}{4} \)

b. \( \binom{10}{3} \)
Discrete Random Variables

Chapter 5

Example 5.13: How many ways can a committee of four students be chosen from a class of 50?

\[ C_{50}^4 = \frac{50!}{4!46!} = \frac{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 46!} = 230,300. \]

It should be clear by this point that there are very many ways to do some everyday tasks that might at first glance seem very mundane and uninteresting.

Example 5.14: How many different groups of 12 fish can decide to split off from the main school of 40 fish?

\[ C_{40}^{12} = \frac{40!}{12!28!} = \frac{40!}{12!28!} = 5.5868535 \times 10^9 = 5,586,853,500. \]

There are over five and one-half billion different groups of 12 fish that could separate from a school of 40 fish!

Example 5.15: A “word” is to be made up of any distinct eight letters of the alphabet (distinct meaning that no letter can be repeated). In how many ways can this be done?

\[ P_{26}^8 = \frac{26!}{(26 - 8)!} = \frac{26!}{18!} = 6.2990928 \times 10^{10} = 62,990,928,000. \]

There are almost sixty-three billion possible arrangements of eight letters without using any letter more than once! This is a permuta-
tion because we are doing more than choosing letters. We are also arranging them.

**Example 5.16:** Now is a good time to return to the California Lottery. Recall that the game is won by picking six numbers correctly from the numbers one to 53. In chapter four we looked at one way to find the probability of winning. Now that we have combinations at our disposal, we have an even simpler way of looking at the problem.

\[
P(\text{win lottery}) = \frac{1}{\binom{53}{6}} = \frac{1}{\frac{53!}{6!47!}} = \frac{1}{22,957,480}
\]

which, of course, is the same answer we obtained before. In words, the formula above simply represents the one way to win out of the \( \binom{53}{6} \approx 23 \) million ways to pick six numbers from 53.

One final note concerning factorials. Occasionally 0! arrises in one of the formulas. Since factorials are usually thought of as the product of numbers down to one some special thought must be given to the meaning of 0!. Since you can't start at 0 and count down to one we must define 0! by other means. 0! is defined to be one. The reason for doing this is that defining 0! to be one makes all of our formulas work as they should. For example, what is \( \binom{10}{10} \)? You don't need the formula to tell the answer! If you have ten objects, how many ways can you choose all ten of them? There is only one way to choose all of them. If we apply the formula we will have:

\[
\binom{10}{10} = \frac{10!}{10!0!} = \frac{1}{0!} = 1.
\]

For this to be one as it should be we must have 0! = 1.
Exercises

Basic Practice

21. Compute each of the following.
   a. $23!$
   b. $\frac{12!}{9!}$
   c. $\frac{12!}{4!(12-4)!}$

22. Compute each of the following.
   a. $\binom{22}{5}$
   b. $\binom{15}{6}$
   c. $23P_8$
   d. $P(48, 19)$
   e. $25C_5$

23. Compute each of the following.
   a. $\binom{16}{9}$
   b. $\binom{16}{5}$
   c. $21P_{16}$
   d. $P(46, 20)$
   e. $19C_{12}$

Applied

24. The 8-th grade class at Howell Mountain Elementary wants to elect class officers. There is to be a president, vice president, treasurer and sergeant-at-arms. If there are 22 students in the class, how many ways can the class officers be picked?
25. How many ways can the four class officers from exercise 24 seat themselves in four chairs?

26. There are 21 kids in Mrs. Grimley’s fourth grade class. How many ways can she pick five kids to help clean up after class?

27. Once Mrs. Grimley (see exercise 26) has picked the five kids to help after class, how many ways can she assign them to the five jobs she has in mind?

28. If you had to pick the six numbers for the California Lottery in the correct order, what would be your probability of winning?

29. A person who owns 50 CD’s wishes to take four with her on a trip.
   a. How many different choices of four discs does she have?
   b. Once she has chosen the four discs, in how many different orders can she play them on the trip?

30. Suppose the President of the United States wants to form a special national defense committee made up of five people. If there are 12 people on his list of candidates, how many ways can he form the committee?

31. The college cafeteria has 23 food items available for lunch. If you are going to compose your lunch of four different items, how many ways can your lunch be chosen?

Concepts

32. Why doesn’t the symbol \( \binom{10}{11} \) make sense? If you did assign a number to \( \binom{10}{11} \) what should it be?

33. What can you learn by comparing \( \binom{12}{4} \) and \( \binom{12}{8} \). Can you suggest a general fact based on your observation?

A Look Back

34. A businessman makes a $1,000 investment with the following possible outcomes and probabilities:
Find the mean and standard deviation of this random variable. If an insured investment earning 4% is available do you think the businessman should have put his money there? Why, or why not?

35. What is the difference between a discrete and continuous random variable?

A Look Ahead

The following exercise anticipates the next section. Try it out without looking there. If you can work it then you know that you won’t have much trouble in the next section. If not, then take a peek ahead and see if you can figure it out.

36. A county jail is “housing” 18 men and four women on a certain day. If four prisoners are chosen at random for a cell check, what is the probability that one will be a woman and three will be men?

5.4 The Hypergeometric Random Variable

Whats Old: This section is based entirely on the combinations introduced in the preceding section.

The hypergeometric probability distribution gives us a tremendous opportunity to exercise our knowledge of combinations in a practical setting. One of the main applications of the hypergeometric distribution involves acceptance sampling. Acceptance sampling is used by production managers to determine if incoming shipments of products meet necessary quality limits.

Before considering an actual acceptance sampling problem we will look at a couple of simpler examples that will illustrate the techniques more easily.

Example 5.17: Suppose a bowl contains ten red marbles and 15 green marbles. If five marbles are drawn at random, what is the probability that exactly two will be red?

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earn $100</td>
<td>0.3</td>
</tr>
<tr>
<td>Earn $20</td>
<td>0.2</td>
</tr>
<tr>
<td>Earn nothing</td>
<td>0.4</td>
</tr>
<tr>
<td>Lose $100</td>
<td>0.1</td>
</tr>
</tbody>
</table>
A picture will help illustrate what we must do.

\[
\begin{align*}
\text{10 red } & \quad \text{choose 2} \\
\text{15 green } & \quad \text{choose 3} \\
\hline
\text{25 total } & \quad \text{choose 5}
\end{align*}
\]

Note above that \(2 + 3 = 5\) and \(10 + 15 = 25\). There is a beautiful order to this arrangement. Now, what formulas are applicable here? If you said “combinations” then either you are picking up the principles of probability quite quickly, or you realize that we learned all those combinations for some reason. Either way the mathematical formula is this:

\[
P(2 \text{ red and 3 green}) = \frac{\binom{10}{2} \binom{15}{3}}{\binom{25}{5}} = 0.38537549
\]

The components of the formula may also be visualized in the following way:

The bottom of the fraction \(\binom{25}{5}\) counts the number of ways that five marbles can be chosen from 25 without restriction. The 10
choose 2 term \( \binom{10}{2} \) counts the number of ways that 2 red marbles can be chosen from 10 and the 15 choose 3 term \( \binom{15}{3} \) counts the number of ways that the 3 remaining green marbles may be chosen from the fifteen.

**Example 5.18:** A farmer has just gathered 32 eggs. Six of the eggs are spoiled. He randomly samples four eggs. What is the probability that exactly one of the four eggs sampled is bad?

\[
P(\text{Exactly one bad egg}) = \frac{\binom{6}{1} \binom{26}{3}}{\binom{32}{4}} = 0.43381535.
\]

Once again, a diagram may aid understanding.

As usual, it is recommended that you work problems using your knowledge of probability rather than by trying to memorize a formula. However, for completeness, the probability function for the hypergeometric distribution is presented.
Suppose there are $k$ objects of type I and $r$ objects of type II. Let $N = k + r$. Suppose $n$ objects are sampled at random from the combined $N$ objects. Let the random variable $X$ be the number of objects of type I that are drawn. Then,

$$P(X = x) = \frac{{k \choose x} {r \choose n-x}}{{N \choose n}}.$$  \hspace{1cm} (5.5)

Of course, for this formula to make sense we need to have $x \leq n$ and $x \leq k$. Otherwise the probability is zero. You can’t draw more objects than you sampled, and you can’t have more objects of type I than there are in the population.

No doubt one look at the above definition will make you a believer in the fact that your best chance lies in understanding rather than memorization. Now it is time for a couple of more practical (and tougher) examples.

**Example 5.19:** The manager of a hardware store chain might wonder if a new shipment of power saws is of acceptable quality. To test the shipment of 50 saws he might randomly select five saws. Assuming that the test results in the two simple possibilities “it works” or “it doesn’t work,” we can use the laws of probability and the formula for combinations to answer a number of practical questions in such a context. For example, let us assume that in actuality four of the 50 saws are bad (don’t work). What is the probability that all of our five sampled saws will work?

$$P(\text{all work}) = \frac{\text{The number of ways we can pick five good saws}}{\text{The total number of ways to pick any five saws}}.$$  

How many ways can we pick five good saws? There are 46 good saws to choose from, so there are

$$\binom{46}{5} = \frac{46!}{5!41!} = 1,370,754$$

ways to do this.
Similarly, without any restrictions there are
\[
\binom{50}{5} = \frac{50!}{5!45!} = 2,118,760
\]
ways to choose five saws.

This tells us that
\[
P(\text{all work}) = \frac{1,370,754}{2,118,760} = 0.64696049.
\]

What is the probability that exactly four of the five work?
\[
P(\text{four work}) = \frac{\binom{46}{4} \binom{4}{1}}{\binom{50}{5}} = \frac{46! \cdot 4!}{4! \cdot 51!45!} = 0.30807642
\]

In the above formula \( \binom{46}{4} \) is the number of ways to choose the four good saws. \( \binom{4}{1} \) is the number of ways to choose the one bad saw. \( \binom{50}{5} \) is the number of ways to choose five saws without restriction.

What is the probability that exactly three of the five work?
\[
P(\text{three work}) = \frac{\binom{46}{3} \binom{4}{2}}{\binom{50}{5}} = \frac{46! \cdot 4!}{3! \cdot 43!2!2! \cdot 50!} = 0.04298741
\]

Of course you could calculate the probabilities for two working and so on, but by now you should have the idea. This type of calculation would be used in practice to decide whether to accept the shipment or not. Of course the manager does not know how many of the saws are defective, but he can still engage in “what if” thinking. How do you think the manager would feel if a sample of five saws only had three saws that worked? From our calculations above we can see that there is a less than five percent chance of a lot with four defective saws yielding a sample of five with three good and two defective saws. The manager would probably return the shipment if two out of five saws were bad.
**Example 5.20:** A quality control engineer samples 15 transformers from a shipment of 60. The shipment is considered acceptable if four or fewer of the 60 transformers are defective. The shipment will be accepted if there is no more than one defective transformer in the sample of 15. What is the probability that a shipment with five bad transformers will be accepted?

\[
P(\text{accept shipment}) = P(0 \text{ or } 1 \text{ defective in sample})
\]

\[
= \frac{\binom{5}{0} \binom{55}{15}}{\binom{60}{15}} + \frac{\binom{5}{1} \binom{55}{40}}{\binom{60}{15}}
\]

\[
= \frac{55!}{15!40!} \binom{60}{15} + \frac{55!}{14!41!} \binom{60}{15}
\]

\[
= 0.22370344 + 0.4092136 = 0.63291704 .
\]

**Exercises**

**Basic Practice**

37. Plug the following \( N, k, x, n \) and \( r \) into the formula for the hypergeometric distribution to calculate the indicated probabilities. Also, say in words what it is that you are calculating in terms of the number of type I objects.
   a. \( P(X = 2) \) for \( N = 25, k = 10, r = 15, n =5 \).
   b. \( P(X = 4) \) for \( N = 50, k = 20, r = 30, n = 10 \).

38. Plug the following \( N, k, x, n \) and \( r \) into the formula for the hypergeometric distribution to calculate the indicated probabilities. Also, say in words what it is that you are calculating in terms of the number of type I objects.
   a. \( P(X = 2) \) for \( N = 35, k = 20, r = 15, n =5 \).
   b. \( P(X = 4) \) for \( N = 50, k = 30, r = 20, n = 10 \).

**Applied**

39. A store has 16 water melons, 12 good and four bad. If you randomly select three water melons from this store, find the probability that two will be good and one will be bad.
40. A personnel manager has 12 employees under her. Five of them are women and seven of them are men. If five people are drawn at random, what is the probability that three of them will be women and two will be men?

41. A quality control engineer samples 12 voltage regulators from a shipment of 50. The shipment is considered acceptable if three or fewer of the regulators are defective. The shipment will be accepted if there is no more than one defective regulator in the sample of 12. What is the probability that a shipment with four bad regulators will be accepted?

42. A car dealership has 40 cars on the lot. Five of them are bombs and will one day be eligible for the “lemon law” (Four consecutive breakdowns of the same part of the car in a short period). If you work for a corporation that keeps a fleet of company cars, and you buy four cars, what is the probability that one of them will be a lemon?

43. A potential jury pool for a drug case contains 20 people, 16 of whom oppose illegal drugs and four who think that drugs should be legalized. If a jury of seven is chosen at random, what is the probability that two of the jurors will think that drugs should be legalized?

44. A politician has 10 good speeches and 20 bad speeches on file. He is going to choose five speeches at random for the next week’s appearances. What is the probability that two of the speeches will be good and three will be bad?

45. Of fifteen unemployed mechanics there are ten who believe 10W-30 oil should be used and five who believe 10W-40 oil should be used. If a new garage hires five of the mechanics, what is the probability that two of them will prefer 10W-30 oil?
5.5 The Binomial Random Variable

**Whats Old:** This section uses the multiplication rule for independent probabilities from chapter four along with combinations to create a probability formula. None of the pieces are new. The only thing new is the formula we build from the existing pieces.

The binomial probability distribution is one of the most applicable of all discrete random variables. As you come to understand it you should be able to think of many situations that are aptly described by this random variable. We set out on the road to understanding via an example.

**Example 5.21:** Suppose an oilman wishes to drill five exploratory oil wells at five different sites. Past experience has taught him that the probability of a successful well is 0.2 and the probability of a dry hole is 0.8. What is the probability that exactly two of the oil wells will be successful.

Once again we do not really need a formula to answer this question. What we already know of probability will be enough. The first question we must answer is “how many ways can two wells be successful and three unsuccessful?” Combinations hold the answer to this question, but before using the formula it might be useful to actually list all the ways that exactly two wells can be successful. To list them we will use the letter \( S \) to represent a successful well and the letter \( U \) to represent an unsuccessful well.

\[
\begin{align*}
SSUUU \\
SUSUU \\
SUUSU \\
SUUUS \\
USSUU \\
USUSU \\
USUUS \\
UUSSU \\
UUSUS \\
UUUSS
\end{align*}
\]

There are ten ways to drill two successful oil wells in five tries. By this we mean that there are ten combinations of two successful sites chosen from five. Using the formula for combinations we can verify this fact.

\[
\binom{5}{2} = \frac{5!}{2!3!} = 10.
\]
Now that we know how many ways that there can be two successful wells we also need to know the probability of occurrence for each of these combinations. Let’s pick on the first one as an example. What is the probability that the first two sites will yield successful wells while the last three sites will yield unsuccessful wells? For each of the wells the probability of a successful well is 0.2 while the probability of an unsuccessful well is 0.8. What we are really asking is “what is the probability the first well will be successful and the second well will be successful and the third unsuccessful and the fourth unsuccessful and the fifth unsuccessful?” Recalling the multiplication rule for probabilities involving the word and shows us that

\[ P(SSUUU) = (0.2)(0.2)(0.8)(0.8)(0.8) \]
\[ = (0.2)^2(0.8)^3 = 0.02048. \]

What is the probability of each of the other ways that the oilman could arrive at two successful wells? They are all the same! The number 0.2 appears twice in each, and the number 0.8 appears three times in each. It doesn’t matter what order we multiply in, so the probability for each of the ten possibilities is the same — 0.02048. Finally, then we have

\[ P(\text{Two successful wells}) = 10(0.2)^2(0.8)^3 = 0.2048. \]

It’s time to put what we learned into the form of a more general formula. The ten ways to have two successful wells came from the process of choosing the two successful drilling sites from the five. This can be done \( \binom{5}{2} \) ways. In general, if we perform an activity \( n \) times the number of ways to have \( x \) successes is \( \binom{n}{x} \). If we let \( p \) represent the probability of success then we will have \( n - x \) failures. So, for the case of \( n \) repetitions we have:
One way to remember what the binomial probability distribution is all about is to perform some linguistics on the word “binomial.” What does “bi” mean? It means “two.” What does “nomial” mean? Just think of the word “nomenclature” which refers to a naming system. “Nomial” just means “name.” So, the binomial distribution is the “two names” distribution. The two names refer to the two possible outcomes — success or failure.

To summarize (and add a couple of details) the following definition is provided.

A binomial random variable has the following characteristics.
1. There are \( n \) identical trials. The probability of success on each trial is denoted by \( p \).
2. Each trial has two possible outcomes. We use the terms “success” and “failure” to identify the two types of outcome.
3. The trials are independent of one another.
4. The binomial random variable \( X \) is the number of successes in \( n \) trials.
5. \( P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \), the probability of \( x \) successes in \( n \) trials.

**Example 5.22:** Joe Cool plans to ask ten girls for a date. (This time he is smart enough to ask them out to different things.) Past experience suggests that he can expect success on twenty percent of his attempts. What is the probability that exactly four girls will say “yes?”
Example 5.23: Joe Cool has gotten a girl to say yes to an invitation to an outdoor concert series. The series consists of ten Monday evening concerts during the summer. It is known that during the summer there is a ten percent chance of rain on any given day. It is also known that two days a week apart are independent of one another as far as rain is concerned. What is the probability that none of the concerts will be rained out?

Remember: \(0! = 1\) and \((0.1)^0 = 1\).

\[
P(X = 4) = \binom{10}{4}(0.2)^4(1 - 0.2)^6 = 0.08808038.
\]

\[
P(X = 0) = \binom{10}{0}(0.1)^0(0.9)^{10} = 1 \cdot (0.9)^{10} \cdot 1 = 0.34867844.
\]

The following box shows the mean (expected value), variance and standard deviation of the binomial random variable.

For a binomial random variable with \(n\) trials and probability of success \(p\):

\[
E(X) = \mu = np
\]

\[
\sigma^2 = np(1 - p)
\]

\[
\sigma = \sqrt{\sigma^2} = \sqrt{np(1 - p)}
\]

Equation (5.6) makes good sense. For example, if you have 100 trials and a probability of success of 0.4 for each trial, you would expect \(np = (0.4)(100) = 40\) success on average. Equations (5.7) and (5.8) are not so easily justified without an understanding of more advanced mathematics and probability theory.

**Comparison Of the Hypergeometric and Binomial Probability Distributions (Optional)**

Now that you have had some exposure to the binomial it’s time to spend a bit more time discussing how to tell the difference between the binomial and hypergeometric. The difference, while simple, is subtle. Both probability models have multiple trials that result in
one of two outcomes. The difference is that in the case of the hypergeometric, items are being selected without replacement from a relatively small population that is made up of just two kinds of things. In the case of the binomial the trials are all independent selections from an unlimited population.

**Example 5.24:** Decide whether each of the following questions is best modeled by a hypergeometric random variable or a binomial random variable.

a. A zoologist has 12 of the 52 eggs laid by the wild whooping crane population. They are to be raised in captivity. If 39 of the 52 eggs are fertile, what is the probability that exactly eight of the 12 eggs that the zoologist has are fertile?

b. A bush pilot making flights in Alaska has records indicating that one in every two thousand flights made by people like himself ends in some type of accident. He figures he will make 500 more flights before he retires. What is the probability that he will have no accidents before he retires?

c. You take a multiple choice Biology test with 50 questions. Each question has four possible choices that are equally likely to be correct. If you guess totally at random, what is the probability that you will get at least 30 questions correct?

In part a the proper model is the hypergeometric model because there are only 52 eggs in existence of which the zoologist has 12. Every time an egg is taken from the wild the remaining population is altered meaning the subsequent eggs taken from the wild will have a different chance of being fertile. This breaks the binomial assumption that every trial has the same probability of success.

In part b the proper model is the binomial model because there is a potentially unlimited number of flights possible.

In part c the proper model is again the binomial model. Although the fact that the test has 50 questions might tempt you to think that a hypergeometric model is appropriate, this is not the case. Actually, the 50 question test itself represents 50 trials from the much larger
(practically unlimited) set of all possible Biology test questions. It is this fact that makes the binomial model appropriate.

Finally, as a convenience, Table 1 in Appendix B is provided for select values of \(n\) and \(p\). It should be used in lieu of the calculator whenever possible simply because it is easier.

## Exercises

### Basic Practice

46. If \(X\) is a binomial random variable with \(n = 15\) and \(p = 0.5\), find \(P(X = 5)\).

47. If \(X\) is a binomial random variable with \(n = 15\) and \(p = 0.05\), find \(P(X = 5)\).

48. Find the mean and variance of the random variable in problem 46.

49. Find the mean and variance of the random variable in problem 47.

50. A die is to be rolled nine times. Find the probability that it will land on “five” exactly three times.

51. Four draws are made with replacement from a jar that contains 10 marbles: seven red and three green. (With replacement means that the marbles are drawn one at a time, and each drawn marble is replaced and the marbles remixed before the next draw.)
   a. What is the probability that all four marbles will be red?
   b. Find the mean \(\mu\) and the standard deviation \(\sigma\) for the number of red marbles drawn.

### Applied

52. A football coach estimates that there is a 0.04 chance that his star halfback will be injured in any game. Assuming that football games are independent of one another, use a binomial probability model to discover the probability that he will be injured twice in the next 16 games that he plays.
53. What is the probability that the halfback in problem 52 will have at least one injury in the next 16 games? (Remember, “at least one” is the complement of “none.”)

54. What is the expected number of injuries for the halfback in problem 52 over the next 16 games that he plays?

55. A stargazer knows that the probability of seeing a shooting star during any randomly selected one minute interval on the night of July 15 at her location is 0.1. If she picks ten one minute intervals at random and observes the sky, what is the probability that she will see exactly two shooting stars?

56. A traveling salesman has probability 0.1 of making a sale at any given call. If he makes 14 independent calls on Tuesday what is the probability that:
   a. He will make exactly two sales?
   b. He will make exactly one sale.
   c. He will make no sales.
   d. He will make more than two sales.

57. An oil exploration company drills 10 independent oil wells in an area where the probability of a successful well is estimated to be 0.3. If this estimate is approximately correct, find the probability (approximately) that five of the wells will be successful.

58. A student takes a 20 question multiple choice test. Each question has four possible responses. What is the probability that he will get at least 14 questions correct totally by guessing?

Concepts

59. Explain in your own words why \( E(X) = \mu = np \) for the binomial probability distribution.

60. What is the purpose of the \( \binom{n}{X} \) factor in the formula for binomial probabilities?
A Look Back

Note: This exercise applies only to those who covered section 5.4 on the hypergeometric distribution.

61. In each of the following situations distinguish whether the hypergeometric or binomial distributions are appropriate. Note: You are not expected to actually solve these. Just identify the correct model.

a. A catcher has a history of allowing one in every 100 pitches to get away from him for a “passed ball.” What model would be appropriate to find the probability that he will be charged with two passed balls in the next 100 pitches?

b. A teacher has a computerized test bank with 20 essay questions and 30 multiple choice questions. If she randomly selects a quiz with five questions, what is the probability that two of the questions will be essay and three multiple choice?

c. A farmer randomly picks five eggs from a basket containing nine chicken eggs and five goose eggs. What would be the appropriate model to calculate the probability that exactly two of the selected eggs are goose eggs?

5.6 The Poisson Random Variable (Optional)

Unfortunately, the theory underlying the Poisson probability distribution is not something that can easily be explained unless you have a substantial mathematical background. The formulas that will be given will consequently not be justified and you will simply have to accept them. Nonetheless, this distribution is both interesting and useful.

The Poisson probability distribution is used to give probabilities regarding the number of occurrences of some type of event in some unit of time or space. Since in this case the formulas must be accepted more than understood we will begin with the definition and follow with examples.
The Poisson Random Variable (Optional)  Section 5.6

Example 5.25: A gas station wants a way to describe how likely it is for various numbers of cars to arrive between 2:00 and 3:00 P.M. They believe a Poisson random variable might fit this situation quite well with \( \mu = 12 \) cars/hour. Assuming they are right, find \( P(X = 15) \), i.e. the probability that 15 cars will come today between 2:00 and 3:00 P.M.

\[
P(X = 15) = \frac{e^{-12}12^{15}}{15!} = 0.07239112.
\]

Example 5.26: The number of telephone calls received at a university switchboard during the afternoon has a Poisson probability distribution with \( \mu = 5 \) calls/minute. What is the probability that during the next five minutes there will be exactly 23 calls?
This question is a bit trickier because the mean is given in terms of calls per minute — that’s one minute. But, the question asked about the number of calls in five minutes. The new mean for the five minute interval is $\mu = 5 \cdot 5 = 25$ calls/5 minutes.

$$P(X = 23) = \frac{e^{-25}25^{23}}{23!} = 0.07634203.$$ 

**Example 5.27:** If the number of pieces of bubble gum in a scoop of bubble gum ice cream has a Poisson probability distribution with a mean of 12 pieces/scoop, then what is the probability that your next spoonful will have exactly one piece of bubble gum in it? (There are eleven spoonfuls of ice cream per scoop.)

To answer this we must once again determine a new mean on the basis of a spoonful rather than a scoop. One way to see what to do is to just look at a simple ratio and proportion.

$$\frac{1}{11} = \frac{\mu_{\text{scoop}}}{\mu_{\text{scoop}}} = \frac{\mu_{\text{scoop}}}{12}$$

Solving this for $\mu_{\text{scoop}}$ yields

$$\mu_{\text{scoop}} = \frac{12}{11} \text{ pieces/spoon}.$$  

Then,

$$e^{-12/11}\left(\frac{12}{11}\right)^{1} \frac{1}{1!} = 0.36644834.$$ 

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**Exercises**

**Basic Practice**

62. If $X$ is a Poisson random variable with $\mu = 2$ find $P(X = 5)$.
63. If $X$ is a Poisson random variable with $\mu = 6$ find $P(X = 5)$.
64. If $X$ is a Poisson random variable with $\mu = 2$ find $P(X = 2)$. 

65. If $X$ is a Poisson random variable with $\mu = 9$ find $P(X = 5)$.

**Applied**

66. Grandma makes the best chocolate chip cookies. After some testing it has been determined that the number of chips per cookie follows a Poisson probability distribution with an average of 12 chocolate chips per cookie.
   a. If you raid a random cookie from her cookie jar, what is the probability that it will have 16 chocolate chips?
   b. If you get two cookies, what is the probability that they will have a total of 23 chocolate chips?

67. There is an average of 2.3 traffic accidents per hour during the 7:00 to 9:00 A.M. rush hour in Big Town. Assuming that the number of accidents follows a Poisson distribution, find:
   a. the probability that there will be no accidents tomorrow during the morning rush hour in Big Town?
   b. the probability that there will be at least one accident tomorrow during the morning rush hour in Big Town?
   (Hint: Remember the complement rule.)

68. For a certain typist’s work, it has been found that the number of mistakes is described by a Poisson random variable with a mean of two mistakes per page. Find the probability that a page of his work will contain at least one mistake.

69. During the hour between 10:00 A.M. and 11:00 A.M. on Tuesday a certain gas station receives an average of $\mu = 7$ customers. What is the probability that they will receive 10 customers during this hour next Tuesday?

70. A summer youth camp has found that there are an average of $\mu = 2$ injuries per week (most of them minor) attributed to the horseback riding program. What is the probability that there will be no injuries in a two week camp cycle?

71. An expensive brand of cotton bond paper is known to have $\mu = 1.7$ blemishes/sheet. Assuming a Poisson distribution of blemishes, what is the probability that a randomly selected sheet will have no more than three blemishes.

72. If a certain type of toad has $\mu = 3$ warts/cm$^2$, what is the probability that a randomly selected toad with 13 cm$^2$ of skin will have exactly 42 warts?
Glossary

**Random Variable:** A *random variable* is a rule which assigns numbers to all the possible outcomes of an experiment.

**Probability function:** A *probability function* is a rule that assigns probabilities to each of the possible outcomes of an experiment.

**Discrete:** A quantity is said to be *discrete* if it can be or conceivably could be counted.

**Continuous:** A quantity that can only be measured and takes on too many values to be counted is said to be *continuous*. Such quantities can take on all values in a range of numbers.

Mean of a Random Variable: The *mean* of a random variable is defined by:

\[ \mu = \sum XP(X) \]

**Standard Deviation of a Random Variable:** The *standard deviation* of a random variable is defined by:

\[ \sigma = \sqrt{\sum (X - \mu)^2 P(X)} \]

**Permutations:** The number of *permutations* of \( r \) objects chosen from \( n \) distinct objects is given by

\[ P(n, r) = \frac{n!}{(n-r)!} \]

**Combinations:** The number of *combinations* of \( r \) objects chosen from \( n \) distinct objects is given by

\[ C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} \]

**Hypergeometric Random Variable:** The formula (on page 210) will not be repeated here. If you need to review the hypergeometric go back and read the whole section. The formula alone will just confuse you.
Binomial Random Variable:

A binomial random variable has the following characteristics.

1. There are \( n \) identical trials. The probability of success on each trial is denoted by \( p \).
2. Each trial has two possible outcomes. We use the terms “success” and “failure” to identify the two types of outcome.
3. The trials are independent of one another.
4. The binomial random variable \( X \) is the number of successes in \( n \) trials.
5. \( P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \), the probability of \( x \) successes in \( n \) trials.
6. \( E(X) = \mu = np \)
7. \( \sigma^2 = np(1-p) \)
8. \( \sigma = \sqrt{\sigma^2} = \sqrt{np(1-p)} \)
Poisson Random Variable:

Only one number (parameter) must be known to describe a Poisson probability distribution. The number that completely describes the Poisson is the mean, \( \mu \). The Poisson random variable \( X \) counts the number of occurrences of some event of interest. If \( X \) is a Poisson random variable, then

\[
P(X = x) = \frac{e^{-\mu} \mu^x}{x!},
\]

the probability of \( x \) occurrences in an interval of time or space.

There are also a couple of assumptions that are necessary for the Poisson distribution to be appropriate.

1. The probability of an occurrence of the event of interest is the same at all times (or points of space).
2. The chance of the event in question happening twice at the same time is nil.
3. Occurrences are independent of one another.