Chapter 1

Set Theory

INTRODUCTION

This chapter treats some of the elementary ideas and concepts of set theory which are necessary for a modern introduction to probability theory.

SETS, ELEMENTS

Any well defined list or collection of objects is called a *set*; the objects comprising the set are called its *elements* or *members*. We write

$$p \in A \quad \text{if } p \text{ is an element in the set } A$$

If every element of A also belongs to a set B, i.e. if $p \in A$ implies $p \in B$, then A is called a *subset* of B or is said to be *contained* in B; this is denoted by

$$A \subset B \quad \text{or} \quad B \supset A$$

Two sets are *equal* if each is contained in the other; that is,

$$A = B \quad \text{if and only if} \quad A \subset B \text{ and } B \subset A$$

The negations of $p \in A$, $A \subset B$ and $A = B$ are written $p \notin A$, $A \not\subset B$ and $A \neq B$ respectively.

We specify a particular set by either listing its elements or by stating properties which characterize the elements of the set. For example,

$$A = \{1, 3, 5, 7, 9\}$$

means A is the set consisting of the numbers 1, 3, 5, 7 and 9; and

$$B = \{x : x \text{ is a prime number, } x < 15\}$$

means that B is the set of prime numbers less than 15.

Unless otherwise stated, all sets under investigation are assumed to be subsets of some fixed set called the *universal set* and denoted (in this chapter) by U. We also use $\emptyset$ to denote the *empty* or *null* set, i.e. the set which contains no elements; this set is regarded as a subset of every other set. Thus for any set A, we have $\emptyset \subset A \subset U$.

Example 1.1: The sets A and B above can also be written as

$$A = \{x : x \text{ is an odd number, } x < 10\} \quad \text{and} \quad B = \{2, 3, 5, 7, 11, 13\}$$

Observe that $9 \in A$ but $9 \notin B$, and $11 \in B$ but $11 \notin A$; whereas $3 \in A$ and $3 \in B$, and $6 \notin A$ and $6 \notin B$.

Example 1.2: We use the following special symbols:

$\mathbf{N}$ = the set of positive integers: $1, 2, 3, \ldots$

$\mathbf{Z}$ = the set of integers: $\ldots, -2, -1, 0, 1, 2, \ldots$

$\mathbf{R}$ = the set of real numbers.

Thus we have $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{R}$.

Example 1.3: *Intervals* on the real line, defined below, appear very often in mathematics. Here a and b are real numbers with $a < b$.

Open interval from a to b $= (a, b) = \{x : a < x < b\}$

Closed interval from a to b $= [a, b] = \{x : a \leq x \leq b\}$

Open-closed interval from a to b $= (a, b] = \{x : a < x \leq b\}$

Closed-open interval from a to b $= [a, b) = \{x : a \leq x < b\}$

The open-closed and closed-open intervals are also called *half-open* intervals.

Example 1.4: In human population studies, the universal set consists of all the people in the world.

Example 1.5: Let $C = \{x : x^2 = 4,\ x \text{ is odd}\}$. Then $C = \emptyset$; that is, C is the empty set.

The following theorem applies.

Theorem 1.1: Let A, B and C be any sets. Then: (i) $A \subset A$; (ii) if $A \subset B$ and $B \subset A$ then $A = B$; and (iii) if $A \subset B$ and $B \subset C$ then $A \subset C$.

We emphasize that $A \subset B$ does not exclude the possibility that $A = B$. However, if $A \subset B$ but $A \neq B$, then we say that A is a *proper subset* of B. (Some authors use the symbol $\subseteq$ for a subset and the symbol $\subset$ only for a proper subset.)

SET OPERATIONS

Let A and B be arbitrary sets. The *union* of A and B, denoted by $A \cup B$, is the set of elements which belong to A or to B:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Here "or" is used in the sense of and/or.

The *intersection* of A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

If $A \cap B = \emptyset$, that is, if A and B do not have any elements in common, then A and B are said to be *disjoint*.

The *difference* of A and B or the *relative complement* of B with respect to A, denoted by $A \setminus B$, is the set of elements which belong to A but not to B:

$$A \setminus B = \{x : x \in A,\ x \notin B\}$$

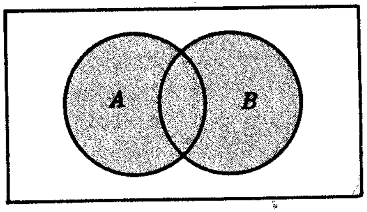
Observe that $A \setminus B$ and B are disjoint, i.e. $(A \setminus B) \cap B = \emptyset$.

The *absolute complement* or, simply, *complement* of A, denoted by A^c, is the set of elements which do not belong to A:

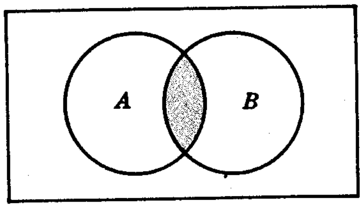
$$A^c = \{x : x \in U,\ x \notin A\}$$

That is, A^c is the difference of the universal set U and A.

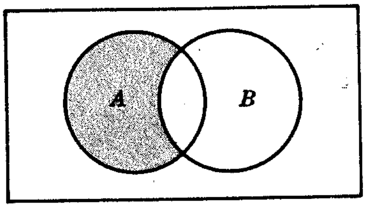
Example 1.6: The following diagrams, called Venn diagrams, illustrate the above set operations. Here sets are represented by simple plane areas and U, the universal set, by the area in the entire rectangle.



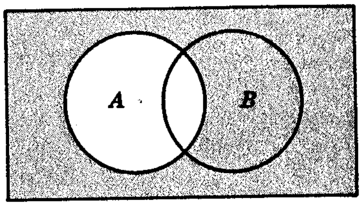
$A \cup B$ is shaded.



$A \cap B$ is shaded.



$A \setminus B$ is shaded.



A^c is shaded.

Example 1.7: Let $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$ where $U = \{1,2,3,\ldots\}$. Then

$$A \cup B = \{1, 2, 3, 4, 5, 6\} \qquad A \cap B = \{3, 4\}$$
$$A \setminus B = \{1, 2\} \qquad A^c = \{5, 6, 7, \ldots\}$$

Sets under the above operations satisfy various laws or identities which are listed in the table below (Table 1). In fact, we state

Theorem 1.2: Sets satisfy the laws in Table 1.

LAWS OF THE ALGEBRA OF SETS	
Idempotent Laws	
1a. $A \cup A = A$	1b. $A \cap A = A$
Associative Laws	
2a. $(A \cup B) \cup C = A \cup (B \cup C)$	2b. $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	
3a. $A \cup B = B \cup A$	3b. $A \cap B = B \cap A$
Distributive Laws	
4a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	4b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws	
5a. $A \cup \emptyset = A$	5b. $A \cap U = A$
6a. $A \cup U = U$	6b. $A \cap \emptyset = \emptyset$
Complement Laws	
7a. $A \cup A^c = U$	7b. $A \cap A^c = \emptyset$
8a. $(A^c)^c = A$	8b. $U^c = \emptyset,\ \emptyset^c = U$
De Morgan's Laws	
9a. $(A \cup B)^c = A^c \cap B^c$	9b. $(A \cap B)^c = A^c \cup B^c$

Table 1

Remark: Each of the above laws follows from an analogous logical law. For example,

$$A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \in A\} = B \cap A$$

Here we use the fact that the composite statement "p and q", written $p \wedge q$, is logically equivalent to the composite statement "q and p", i.e. $q \wedge p$.

The relationship between set inclusion and the above set operations follows:

Theorem 1.3: Each of the following conditions is equivalent to $A \subset B$:

(i) $A \cap B = A$ (iii) $B^c \subset A^c$ (v) $B \cup A^c = U$

(ii) $A \cup B = B$ (iv) $A \cap B^c = \emptyset$

FINITE AND COUNTABLE SETS

Sets can be finite or infinite. A set is finite if it is empty or if it consists of exactly n elements where n is a positive integer; otherwise it is infinite.

Example 1.8: Let M be the set of the days of the week; that is,

$$M = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}\}$$

Then M is finite.

Example 1.9: Let $P = \{x : x \text{ is a river on the earth}\}$. Although it may be difficult to count the number of rivers on the earth, P is a finite set.

Example 1.10: Let Y be the set of (positive) even integers, i.e. $Y = \{2, 4, 6, \ldots\}$. Then Y is an infinite set.

Example 1.11: Let I be the *unit interval* of real numbers, i.e. $I = \{x : 0 \leq x \leq 1\}$. Then I is also an infinite set.

A set is *countable* if it is finite or if its elements can be arranged in the form of a sequence, in which case it is said to be *countably infinite*; otherwise the set is *uncountable*. The set in Example 1.10 is countably infinite, whereas it can be shown that the set in Example 1.11 is uncountable.

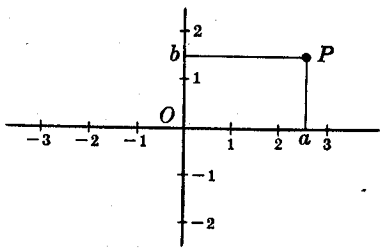
PRODUCT SETS

Let A and B be two sets. The *product set* of A and B, denoted by $A \times B$, consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) : a \in A, \ b \in B\}$$

The product of a set with itself, say $A \times A$, is denoted by A^2.

Example 1.12: The reader is familiar with the cartesian plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ as shown below. Here each point P represents an ordered pair (a, b) of real numbers, and vice versa.



Example 1.13: Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

The concept of product set is extended to any finite number of sets in a natural way. The product set of the sets $A_1, A_2, \ldots, A_m$, written $A_1 \times A_2 \times \cdots \times A_m$, is the set of all ordered m-tuples $(a_1, a_2, \ldots, a_m)$ where $a_i \in A_i$ for each i.

CLASSES OF SETS

Frequently the members of a set are sets themselves. For example, each line in a set of lines is a set of points. To help clarify these situations, we usually use the word *class* or *family* for such a set. The words subclass and subfamily have meanings analogous to subset.

Example 1.14: The members of the class $\{\{2,3\}, \{2\}, \{5,6\}\}$ are the sets $\{2,3\}$, $\{2\}$ and $\{5,6\}$.

Example 1.15: Consider any set A. The *power set* of A, denoted by $\mathcal{P}(A)$, is the class of all subsets of A. In particular, if $A = \{a, b, c\}$, then

$$\mathcal{P}(A) = \{A, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$$

In general, if A is finite and has n elements, then $\mathcal{P}(A)$ will have 2^n elements.

A *partition* of a set X is a subdivision of X into nonempty subsets which are disjoint and whose union is X, i.e. is a class of nonempty subsets of X such that each $a \in X$ belongs to a unique subset. The subsets in a partition are called *cells*.

Example 1.16: Consider the following classes of subsets of $X = \{1, 2, \ldots, 8, 9\}$:

(i) $[\{1,3,5\}, \{2,6\}, \{4,8,9\}]$

(ii) $[\{1,3,5\}, \{2,4,6,8\}, \{5,7,9\}]$

(iii) $[\{1,3,5\}, \{2,4,6,8\}, \{7,9\}]$

Then (i) is not a partition of X since $7 \in X$ but 7 does not belong to any of the cells. Furthermore, (ii) is not a partition of X since $5 \in X$ and 5 belongs to both $\{1,3,5\}$ and $\{5,7,9\}$. On the other hand, (iii) is a partition of X since each element of X belongs to exactly one cell.

When we speak of an *indexed class of sets* $\{A_i : i \in I\}$ or simply $\{A_i\}$, we mean that there is a set A_i assigned to each element $i \in I$. The set I is called the *indexing set* and the sets A_i are said to be indexed by I. When the indexing set is the set $\mathbf{N}$ of positive integers, the indexed class $\{A_1, A_2, \ldots\}$ is called a *sequence* of sets. By the *union* of these A_i, denoted by $\bigcup_{i \in I} A_i$ (or simply $\bigcup_i A_i$), we mean the set of elements each belonging to at least one of the A_i; and by the *intersection* of the A_i, denoted by $\bigcap_{i \in I} A_i$ (or simply $\bigcap_i A_i$), we mean the set of elements each belonging to every A_i. We also write

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots \quad \text{and} \quad \bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots$$

for the union and intersection, respectively, of a sequence of sets.

Definition: A nonempty class $\mathcal{A}$ of subsets of U is called an *algebra* (σ-*algebra*) of sets if:

(i) the complement of any set in $\mathcal{A}$ belongs to $\mathcal{A}$; and

(ii) the union of any finite (countable) number of sets in $\mathcal{A}$ belongs to $\mathcal{A}$;

that is, if $\mathcal{A}$ is closed under complements and finite (countable) unions.

It is simple to show (Problem 1.30) that an algebra (σ-algebra) of sets contains U and $\emptyset$ and is also closed under finite (countable) intersections.

Solved Problems

SETS, ELEMENTS, SUBSETS

1.1. Let $A = \{x : 3x = 6\}$. Does $A = 2$?

A is the set which consists of the single element 2, that is, $A = \{2\}$. The number 2 belongs to A; it does not equal A. There is a basic difference between an element p and the singleton set $\{p\}$.

1.2. Which of these sets are equal: $\{r, s, t\}$, $\{t, s, r\}$, $\{s, r, t\}$, $\{t, r, s\}$?

They are all equal. Order does not change a set.

1.3. Determine whether or not each set is the null set:

(i) $X = \{x : x^2 = 9,\ 2x = 4\}$, (ii) $Y = \{x : x \neq x\}$, (iii) $Z = \{x : x + 8 = 8\}$.

(i) There is no number which satisfies both $x^2 = 9$ and $2x = 4$; hence X is empty, i.e. $X = \emptyset$.

(ii) We interpret "=" to mean "is identical with" and so Y is also empty. In fact, some texts define the empty set as follows: $\emptyset \equiv \{x : x \neq x\}$.

(iii) The number zero satisfies $x + 8 = 8$; hence $Z = \{0\}$. Accordingly, Z is not the empty set since it contains 0. That is, $Z \neq \emptyset$.

1.4. Prove that $A = \{2, 3, 4, 5\}$ is not a subset of $B = \{x : x \text{ is even}\}$.

It is necessary to show that at least one element in A does not belong to B. Now $3 \in A$ and, since B consists of even numbers, $3 \notin B$; hence A is not a subset of B.

1.5. Let $V = \{d\}$, $W = \{c, d\}$, $X = \{a, b, c\}$, $Y = \{a, b\}$ and $Z = \{a, b, d\}$. Determine whether each statement is true or false:

(i) $Y \subset X$, (ii) $W \neq Z$, (iii) $Z \supset V$, (iv) $V \subset X$, (v) $X = W$, (vi) $W \subset Y$.

(i) Since each element in Y is a member of X, $Y \subset X$ is true.

(ii) Now $a \in Z$ but $a \notin W$; hence $W \neq Z$ is true.

(iii) The only element in V is d and it also belongs to Z; hence $Z \supset V$ is true.

(iv) V is not a subset of X since $d \in V$ but $d \notin X$; hence $V \subset X$ is false.

(v) Now $a \in X$ but $a \notin W$; hence $X = W$ is false.

(vi) W is not a subset of Y since $c \in W$ but $c \notin Y$; hence $W \subset Y$ is false.

1.6. Prove: If A is a subset of the empty set $\emptyset$, then $A = \emptyset$.

The null set $\emptyset$ is a subset of every set; in particular, $\emptyset \subset A$. But, by hypothesis, $A \subset \emptyset$; hence $A = \emptyset$.

1.7. Prove Theorem 1.1(iii): If $A \subset B$ and $B \subset C$, then $A \subset C$.

We must show that each element in A also belongs to C. Let $x \in A$. Now $A \subset B$ implies $x \in B$. But $B \subset C$; hence $x \in C$. We have shown that $x \in A$ implies $x \in C$, that is, that $A \subset C$.

1.8. Which of the following sets are finite?

(i) The months of the year.

(ii) $\{1, 2, 3, \ldots, 99, 100\}$.

(iii) The number of people living on the earth.

(iv) The set **Q** of rational numbers.

(v) The set **R** of real numbers.

The first three sets are finite; the last two are infinite. (It can be shown that **Q** is countable but **R** is uncountable.)

1.9. Consider the following sets of figures in the Euclidean plane:

$$A = \{x : x \text{ is a quadrilateral}\} \qquad C = \{x : x \text{ is a rhombus}\}$$
$$B = \{x : x \text{ is a rectangle}\} \qquad D = \{x : x \text{ is a square}\}$$

Determine which sets are proper subsets of any of the others.

Since a square has 4 right angles it is a rectangle, since it has 4 equal sides it is a rhombus, and since it has 4 sides it is a quadrilateral. Thus

$$D \subset A, \quad D \subset B \quad \text{and} \quad D \subset C$$

that is, D is a subset of the other three. Also, since there are examples of rectangles, rhombuses and quadrilaterals which are not squares, D is a proper subset of the other three.

In a similar manner we see that B is a proper subset of A and C is a proper subset of A. There are no other relations among the sets.

1.10. Determine which of the following sets are equal: $\emptyset$, $\{0\}$, $\{\emptyset\}$.

Each is different from the other. The set $\{0\}$ contains one element, the number zero. The set $\emptyset$ contains no elements; it is the empty set. The set $\{\emptyset\}$ also contains one element, the null set.

SET OPERATIONS

1.11. Let $U = \{1, 2, \ldots, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Find: (i) A^c, (ii) $A \cap C$, (iii) $(A \cap C)^c$, (iv) $A \cup B$, (v) $B \setminus C$.

(i) A^c consists of the elements in U that are not in A; hence $A^c = \{5, 6, 7, 8, 9\}$.

(ii) $A \cap C$ consists of the elements in both A and C; hence $A \cap C = \{3, 4\}$.

(iii) $(A \cap C)^c$ consists of the elements in U that are not in $A \cap C$. Now by (ii), $A \cap C = \{3, 4\}$ and so $(A \cap C)^c = \{1, 2, 5, 6, 7, 8, 9\}$.

(iv) $A \cup B$ consists of the elements in A or B (or both): hence $A \cup B = \{1, 2, 3, 4, 6, 8\}$.

(v) $B \setminus C$ consists of the elements in B which are not in C; hence $B \setminus C = \{2, 8\}$.

1.12. Let $U = \{a, b, c, d, e\}$, $A = \{a, b, d\}$ and $B = \{b, d, e\}$. Find:

(i) $A \cup B$ (iii) B^c (v) $A^c \cap B$ (vii) $A^c \cap B^c$ (ix) $(A \cap B)^c$

(ii) $B \cap A$ (iv) $B \setminus A$ (vi) $A \cup B^c$ (viii) $B^c \setminus A^c$ (x) $(A \cup B)^c$

(i) The union of A and B consists of the elements in A or in B (or both); hence $A \cup B = \{a, b, d, e\}$.

(ii) The intersection of A and B consists of those elements which belong to both A and B; hence $A \cap B = \{b, d\}$.

(iii) The complement of B consists of the letters in U but not in B; hence $B^c = \{a, c\}$.

(iv) The difference $B \setminus A$ consists of the elements of B which do not belong to A; hence $B \setminus A = \{e\}$.

(v) $A^c = \{c, e\}$ and $B = \{b, d, e\}$; then $A^c \cap B = \{e\}$.

(vi) $A = \{a, b, d\}$ and $B^c = \{a, c\}$; then $A \cup B^c = \{a, b, c, d\}$.

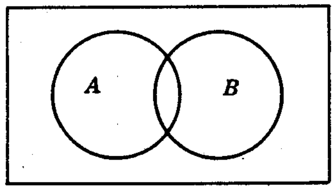
(vii) and (viii). $A^c = \{c, e\}$ and $B^c = \{a, c\}$; then

$$A^c \cap B^c = \{c\} \quad \text{and} \quad B^c \setminus A^c = \{a\}$$

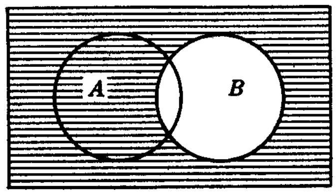
(ix) From (ii), $A \cap B = \{b, d\}$; hence $(A \cap B)^c = \{a, c, e\}$.

(x) From (i), $A \cup B = \{a, b, d, e\}$; hence $(A \cup B)^c = \{c\}$.

1.13. In the Venn diagram below, shade: (i) B^c, (ii) $(A \cup B)^c$, (iii) $(B \setminus A)^c$, (iv) $A^c \cap B^c$.

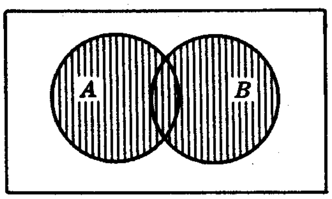


(i) B^c consists of the elements which do not belong to B; hence shade the area outside B as follows:

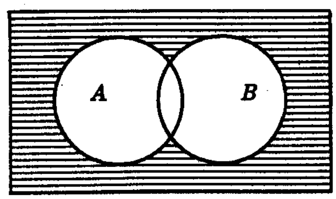


B^c is shaded.

(ii) First shade $A \cup B$; then $(A \cup B)^c$ is the area outside $A \cup B$:

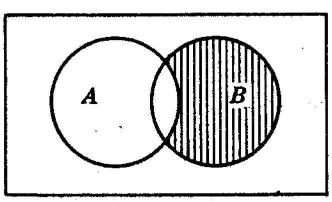


$A \cup B$ is shaded.

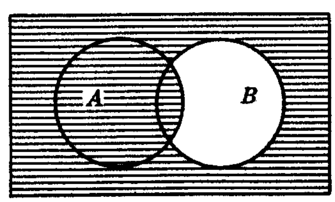


$(A \cup B)^c$ is shaded.

(iii) First shade $B \setminus A$, the area in B which does not lie in A; then $(B \setminus A)^c$ is the area outside $B \setminus A$:

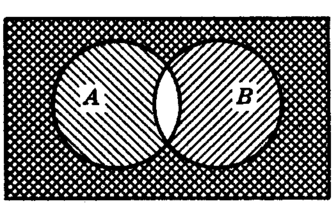


$B \setminus A$ is shaded.

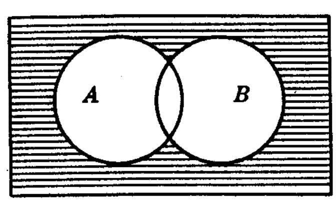


$(B \setminus A)^c$ is shaded.

(iv) First shade A^c, the area outside of A, with strokes slanting upward to the right (////), and then shade B^c with strokes slanting downward to the right (\\\\\\\\); then $A^c \cap B^c$ is the cross-hatched area:



A^c and B^c are shaded.



$A^c \cap B^c$ is shaded.

Observe that $(A \cup B)^c = A^c \cap B^c$, as expected by De Morgan's law.

1.14. Prove: $B \setminus A = B \cap A^c$. Thus the set operation of difference can be written in terms of the operations of intersection and complementation.

$$B \setminus A \quad = \quad \{x : x \in B,\ x \notin A\} \quad = \quad \{x : x \in B,\ x \in A^c\} \quad = \quad B \cap A^c$$

1.15. Prove: For any sets A and B, $A \cap B \subset A \subset A \cup B$.

Let $x \in A \cap B$; then $x \in A$ and $x \in B$. In particular, $x \in A$. Since $x \in A \cap B$ implies $x \in A$, $A \cap B \subset A$. Furthermore if $x \in A$, then $x \in A$ or $x \in B$, i.e. $x \in A \cup B$. Hence $A \subset A \cup B$. In other words, $A \cap B \subset A \subset A \cup B$.

1.16. Prove Theorem 1.3(i): $A \subset B$ if and only if $A \cap B = A$.

Suppose $A \subset B$. Let $x \in A$; then by hypothesis, $x \in B$. Hence $x \in A$ and $x \in B$, i.e. $x \in A \cap B$. Accordingly, $A \subset A \cap B$. On the other hand, it is always true (Problem 1.15) that $A \cap B \subset A$. Thus $A \cap B = A$.

Now suppose that $A \cap B = A$. Then in particular, $A \subset A \cap B$. But it is always true that $A \cap B \subset B$. Thus $A \subset A \cap B \subset B$ and so, by Theorem 1.1, $A \subset B$.

PRODUCT SETS

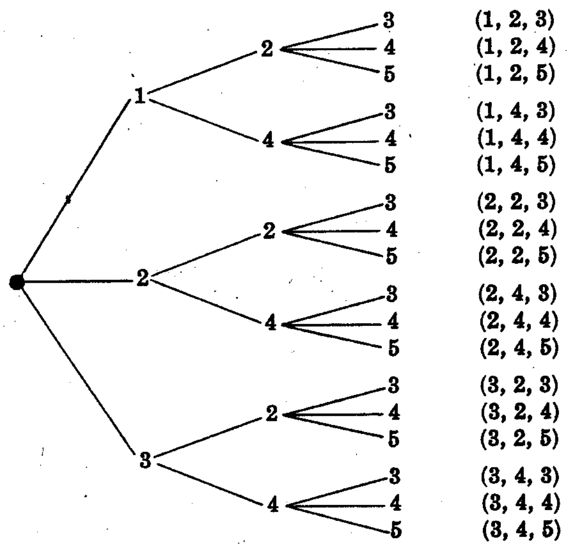
1.17. Let M = {Tom, Marc, Erik} and W = {Audrey, Betty}. Find $M \times W$.

$M \times W$ consists of all ordered pairs (a, b) where $a \in M$ and $b \in W$. Hence

$M \times W$ = {(Tom, Audrey), (Tom, Betty), (Marc, Audrey), (Marc, Betty), (Erik, Audrey), (Erik, Betty)}

1.18. Let $A = \{1, 2, 3\}$, $B = \{2, 4\}$ and $C = \{3, 4, 5\}$. Find $A \times B \times C$.

A convenient method of finding $A \times B \times C$ is through the so-called "tree diagram" shown below:



The "tree" is constructed from the left to the right. $A \times B \times C$ consists of the ordered triples listed to the right of the "tree".

1.19. Let $A = \{a, b\}$, $B = \{2, 3\}$ and $C = \{3, 4\}$. Find:
(i) $A \times (B \cup C)$, (ii) $(A \times B) \cup (A \times C)$, (iii) $A \times (B \cap C)$, (iv) $(A \times B) \cap (A \times C)$.

(i) First compute $B \cup C = \{2, 3, 4\}$. Then

$$A \times (B \cup C) \quad = \quad \{(a, 2), (a, 3), (a, 4), (b, 2), (b, 3), (b, 4)\}$$

(ii) First find $A \times B$ and $A \times C$:

$$A \times B = \{(a,2), (a,3), (b,2), (b,3)\}$$
$$A \times C = \{(a,3), (a,4), (b,3), (b,4)\}$$

Then compute the union of the two sets:

$$(A \times B) \cup (A \times C) = \{(a,2), (a,3), (b,2), (b,3), (a,4), (b,4)\}$$

Observe from (i) and (ii) that

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

(iii) First compute $B \cap C = \{3\}$. Then

$$A \times (B \cap C) = \{(a,3), (b,3)\}$$

(iv) Now $A \times B$ and $A \times C$ were computed above. The intersection of $A \times B$ and $A \times C$ consists of those ordered pairs which belong to both sets:

$$(A \times B) \cap (A \times C) = \{(a,3), (b,3)\}$$

Observe from (iii) and (iv) that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

1.20. Prove: $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

$$\begin{aligned} A \times (B \cap C) &= \{(x,y) : x \in A,\ y \in B \cap C\} \\ &= \{(x,y) : x \in A,\ y \in B,\ y \in C\} \\ &= \{(x,y) : (x,y) \in A \times B,\ (x,y) \in A \times C\} \\ &= (A \times B) \cap (A \times C) \end{aligned}$$

1.21. Let $S = \{a,b\}$, $W = \{1,2,3,4,5,6\}$ and $V = \{3,5,7,9\}$. Find $(S \times W) \cap (S \times V)$.

The product set $(S \times W) \cap (S \times V)$ can be found by first computing $S \times W$ and $S \times V$, and then computing the intersection of these sets. On the other hand, by the preceding problem, $(S \times W) \cap (S \times V) = S \times (W \cap V)$. Now $W \cap V = \{3,5\}$, and so

$$(S \times W) \cap (S \times V) = S \times (W \cap V) = \{(a,3), (a,5), (b,3), (b,5)\}$$

1.22. Prove: Let $A \subset B$ and $C \subset D$; then $(A \times C) \subset (B \times D)$.

Let (x,y) be any arbitrary element in $A \times C$; then $x \in A$ and $y \in C$. By hypothesis, $A \subset B$ and $C \subset D$; hence $x \in B$ and $y \in D$. Accordingly (x,y) belongs to $B \times D$. We have shown that $(x,y) \in A \times C$ implies $(x,y) \in B \times D$; hence $(A \times C) \subset (B \times D)$.

CLASSES OF SETS

1.23. Consider the class $A = \{\{2,3\}, \{4,5\}, \{6\}\}$. Which statements are incorrect and why? (i) $\{4,5\} \subset A$, (ii) $\{4,5\} \in A$, (iii) $\{\{4,5\}\} \subset A$.

The members of A are the sets $\{2,3\}$, $\{4,5\}$ and $\{6\}$. Therefore (ii) is correct but (i) is an incorrect statement. Moreover, (iii) is also a correct statement since the set consisting of the single element $\{4,5\}$ is a subclass of A.

1.24. Find the power set $\mathcal{P}(S)$ of the set $S = \{1,2,3\}$.

The power set $\mathcal{P}(S)$ of S is the class of all subsets of S; these are $\{1,2,3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1\}$, $\{2\}$, $\{3\}$ and the empty set $\emptyset$. Hence

$$\mathcal{P}(S) = \{S, \{1,3\}, \{2,3\}, \{1,2\}, \{1\}, \{2\}, \{3\}, \emptyset\}$$

Note that there are $2^3 = 8$ subsets of S.

1.25. Let $X = \{a, b, c, d, e, f, g\}$, and let:

(i) $A_1 = \{a, c, e\}$, $A_2 = \{b\}$, $A_3 = \{d, g\}$;

(ii) $B_1 = \{a, e, g\}$, $B_2 = \{c, d\}$, $B_3 = \{b, e, f\}$;

(iii) $C_1 = \{a, b, e, g\}$, $C_2 = \{c\}$, $C_3 = \{d, f\}$;

(iv) $D_1 = \{a, b, c, d, e, f, g\}$.

Which of $\{A_1, A_2, A_3\}$, $\{B_1, B_2, B_3\}$, $\{C_1, C_2, C_3\}$, $\{D_1\}$ are partitions of X?

(i) $\{A_1, A_2, A_3\}$ is not a partition of X since $f \in X$ but f does not belong to either A_1, A_2, or A_3.

(ii) $\{B_1, B_2, B_3\}$ is not a partition of X since $e \in X$ belongs to both B_1 and B_3.

(iii) $\{C_1, C_2, C_3\}$ is a partition of X since each element in X belongs to exactly one cell, i.e. $X = C_1 \cup C_2 \cup C_3$ and the sets are pairwise disjoint.

(iv) $\{D_1\}$ is a partition of X.

1.26. Find all the partitions of $X = \{a, b, c, d\}$.

Note first that each partition of X contains either 1, 2, 3, or 4 distinct sets. The partitions are as follows:

(1) $[\{a, b, c, d\}]$

(2) $[\{a\}, \{b, c, d\}]$, $[\{b\}, \{a, c, d\}]$, $[\{c\}, \{a, b, d\}]$, $[\{d\}, \{a, b, c\}]$,
$[\{a, b\}, \{c, d\}]$, $[\{a, c\}, \{b, d\}]$, $[\{a, d\}, \{b, c\}]$

(3) $[\{a\}, \{b\}, \{c, d\}]$, $[\{a\}, \{c\}, \{b, d\}]$, $[\{a\}, \{d\}, \{b, c\}]$,
$[\{b\}, \{c\}, \{a, d\}]$, $[\{b\}, \{d\}, \{a, c\}]$, $[\{c\}, \{d\}, \{a, b\}]$

(4) $[\{a\}, \{b\}, \{c\}, \{d\}]$

There are fifteen different partitions of X.

1.27. Let $\mathbf{N}$ be the set of positive integers and, for each $n \in \mathbf{N}$, let

$$A_n = \{x : x \text{ is a multiple of } n\} = \{n, 2n, 3n, \ldots\}$$

Find (i) $A_3 \cap A_5$, (ii) $A_4 \cap A_6$, (iii) $\cup_{i \in P} A_i$, where P is the set of prime numbers, 2, 3, 5, 7, 11, ...

(i) Those numbers which are multiples of both 3 and 5 are the multiples of 15; hence $A_3 \cap A_5 = A_{15}$.

(ii) The multiples of 12 and no other numbers belong to both A_4 and A_6; hence $A_4 \cap A_6 = A_{12}$.

(iii) Every positive integer except 1 is a multiple of at least one prime number; hence

$$\cup_{i \in P} A_i = \{2, 3, 4, \ldots\} = \mathbf{N} \setminus \{1\}$$

1.28. Prove: Let $\{A_i : i \in I\}$ be an indexed class of sets and let $i_0 \in I$. Then

$$\cap_{i \in I} A_i \subset A_{i_0} \subset \cup_{i \in I} A_i$$

Let $x \in \cap_{i \in I} A_i$; then $x \in A_i$ for every $i \in I$. In particular, $x \in A_{i_0}$. Hence $\cap_{i \in I} A_i \subset A_{i_0}$. Now let $y \in A_{i_0}$. Since $i_0 \in I$, $y \in \cup_{i \in I} A_i$. Hence $A_{i_0} \subset \cup_{i \in I} A_i$.

1.29. Prove (De Morgan's law): For any indexed class $\{A_i : i \in I\}$, $(\cup_i A_i)^c = \cap_i A_i^c$.

$$(\cup_i A_i)^c = \{x : x \notin \cup_i A_i\} = \{x : x \notin A_i \text{ for every } i\} = \{x : x \in A_i^c \text{ for every } i\} = \cap_i A_i^c$$

1.30. Let $\mathcal{A}$ be an algebra (σ-algebra) of subsets of U. Show that: (i) U and $\emptyset$ belong to $\mathcal{A}$; and (ii) $\mathcal{A}$ is closed under finite (countable) intersections.

Recall that $\mathcal{A}$ is closed under complements and finite (countable) unions.

(i) Since $\mathcal{A}$ is nonempty, there is a set $A \in \mathcal{A}$. Hence the complement $A^c \in \mathcal{A}$, and the union $U = A \cup A^c \in \mathcal{A}$. Also the complement $\emptyset = U^c \in \mathcal{A}$.

(ii) Let $\{A_i\}$ be a finite (countable) class of sets belonging to $\mathcal{A}$. By De Morgan's law (Problem 1.29), $(\cup_i A_i^c)^c = \cap_i A_i^{cc} = \cap_i A_1$. Hence $\cap_i A_i$ belongs to $\mathcal{A}$, as required.